Solution to Discussion Problem from September 19, 2014

Daniel McLaury

For those of you playing along at home (which, in theory, should be all of you), here's a complete solution to the problem put up on the board in class. Try to work it out yourself, but you can use this to check your answer or if you get stuck. Additional explanation of the solution on my part is marked in red; everything else should show up in your solution if you were answering a question like this on a homework set or exam.

Problem

Sketch the graph of the function

$$f(x) = \frac{x^2}{\sqrt{x^2+1}(x-1)}.$$

Justify all features of the graph.

Solution

To start with, let's determine end behavior. We have

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2}{\sqrt{x^2+1}(x-1)} \div \frac{x^2}{x^2} = \lim_{x\to-\infty} \frac{1}{\frac{\sqrt{x^2+1}}{x}\cdot\frac{x-1}{x}} = \lim_{x\to-\infty} \frac{1}{\frac{\sqrt{x^2+1}}{x}\cdot\left(1-\frac{1}{x}\right)}$$

We want to bring that x down there up into the square root, cancelling the x^2, but we need to be careful about the sign. $\sqrt{x^2} = |x|$; for instance, if we take -3 and square it, we get 9; taking the square root of that gives us 3. One way to deal with this is to multiply by $|x|/\sqrt{x^2}$, which is equal to 1 (at least so long as $x \neq 0$). Now

$$\frac{\sqrt{x^2+1}}{x} = \frac{|x|}{\sqrt{x^2}}\frac{\sqrt{x^2+1}}{x} = \frac{|x|}{x}\sqrt{1+\frac{1}{x^2}}.$$

Notice that

$$\frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0; \end{cases}$$

sometimes people also call this function $\operatorname{sign}(x)$ for this reason. Putting this back into the limit, we have

$$= \lim_{x\to\infty} \frac{1}{\operatorname{sign}(x)\sqrt{1+\cancelto{0}{\frac{1}{x^2}}}\left(1+\cancelto{0}{\frac{1}{x}}\right)} = \frac{1}{-1\cdot 1\cdot 1} = -1.$$

Similarly,

$$\lim_{x\to\infty} f(x) = 1,$$

so there are horizontal asymptotes at $y=-1$ and $y=1$.

Okay, let's do vertical asymptotes now. The denominator vanishes at $x = 1$ and the numerator doesn't, so there's a vertical asymptote at $x = 1$. Since $f(x) > 0$ for $x > 1$ and $f(x) < 0$ for $x < 1$, we have

$$\lim_{x\to 1^-} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to 1^+} f(x) = +\infty.$$

The denominator doesn't have any other zeroes, so there are no other vertical asymptotes.

Now let's work out where the function is increasing/decreasing, whether there are any local minima/maxima, etc. We have

$$f'(x) = \frac{\sqrt{x^2+1}(x-1)(2x) - x^2\left[\sqrt{x^2+1}(1) + \frac{1}{2}(x^2+1)^{-1/2}(2x)(x-1)\right]}{(x^2+1)(x-1)^2}.$$

(multiply the top and bottom by $\sqrt{x^2+1}$ to simplify)

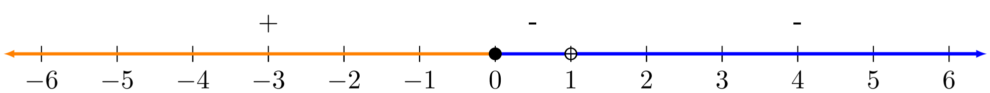
$$= \frac{(x^2+1)(x-1)(2x) - x^2\left[(x^2+1)(1) + \frac{1}{2}(2x)(x-1)\right]}{(x^2+1)^{3/2}(x-1)^2}$$

$$= \frac{(x^3-x^2+x-1)(2x) - x^2[(x^2+1)+(x^2-x)]}{(x^2+1)^{3/2}(x-1)^2}$$

$$= \frac{2x^4-2x^3+2x^2-2x-(2x^4-x^3+x^2)}{(x^2+1)^{3/2}(x-1)^2}$$

$$= \frac{-x^3+x^2-2x}{(x^2+1)^{3/2}(x-1)^2} = \frac{-x(x^2-x+2)}{(x^2+1)^{3/2}(x-1)^2}$$

The derivative is undefined at $x = 1$ and zero at $x = 0$. We know that $x^2 - x + 2$ is never zero, so we don't have to worry about any zeroes coming from there. (You could determine this by the quadratic formula / completing the square, for instance.) So the sign of $f'(x)$ is



Therefore the function has a local maximum at $x = 0$ and approaches both horizontal asymptotes from above, like so: